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# THE MONOTONICITY PROPERTIES OF THE SOLUTIONS OF FIRST-ORDER ELLIPTIC SYSTEMS AND THEIR APPLICATIONS TO THE EQUATIONS OF FLUID AND GAS MECHANICS†

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**It is shown that the solution of a homogeneous quasilinear elliptic system of two equations has a monotonicity property, according to which each of the two functions considered varies monotonically along the level line of the other function. A connection between the monotonicity property and the extremal properties of the solutions of the systems considered is pointed out. An algorithm for converting certain inhomogeneous and, in particular, homogeneous systems into homogeneous ones is proposed, which considerably extends the amount of information on the monotonicity properties and on the extremal properties of the solutions. The value of the results obtained is demonstrated using the example of the axisymmetric and plane flow of an incompressible fluid and a gas.**

Monotonicity properties were established for the first time for plane subsonic potential flows of a gas and later also for vortex flows [1, 2], where the modulus of the velocity vector (later the pressure) and the slope of the velocity vector were used as the functions. This property has been widely employed [1–7] to analyse potential and vortex flows. The monotonicity property was also established in [8] for the asymptotic equations of transonic axisymmetric subsonic flows.

In this paper we initiate a natural trend, on the one hand, to investigate how wide is the class of systems the solutions of which possess the monotonicity property and, on the other hand, to widen the amount of information on the monotonicity properties of each of the systems considered, in particular, for the equations of hydrodynamics.

1. Consider the following homogeneous quasilinear elliptic system

$$a_{i1}u_x + a_{i2}u_y + b_{i1}\upsilon_x + b_{i2}\upsilon_y = 0 \quad (i = 1,2) \tag{1.1}$$

We will assume that $a_{ij} = a_{ij}(x, y, u, \upsilon)$, $b_{ij} = b_{ij}(x, y, u, \upsilon)$ are fairly continuous and bounded functions of their arguments.

Many problems in mechanics and physics reduce to an investigation of systems of the form (1.1), and also systems of the form (2.1) which will be considered below.

Using the matrices $A = (a_{ij})$ and $B = (b_{ij})$, system (1.1) can be written as

$$A\nabla u + B\nabla \upsilon = 0$$

By definition, system (1.1) is elliptic if the characteristic determinant

$$h(\varphi) = \begin{vmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ \cos\varphi & \sin\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & \sin\varphi \end{vmatrix}$$

does not vanish in the region considered for any real values of the angle $\varphi$.

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Together with the determinant $h$, we will henceforth use the determinants $|A|$, $|B|$ of matrices $A$ and $B$. It can be shown, and this will be important later, that these determinants do not vanish for elliptic system (1.1).

In fact, an analysis of the discriminant of the characteristic equation $h(\varphi) = 0$ shows that

$$d = (a_{11}b_{22} - a_{21}b_{12} + a_{22}b_{11} - a_{12}b_{21})^2 - 4|A|\,|B|$$

In the region in which system (1.1) is elliptic, $d < 0$. Consequently, in this region $|A| \neq 0$, $|B| \neq 0$, $|A|\,|B| > 0$.

Consider the curve $u = \text{const}$, which is the boundary of the chosen region of an increased or reduced value of $u$ (with respect to the curve $u = \text{const}$). This means that when passing through a possible branching point, to extend the curve $u = \text{const}$ the branch adjoining the given region is chosen. As a result, when moving along this curve the sign of the normal derivative of $u_n$ does not change.

*Theorem* 1. When moving along the curve $u = \text{const}$ the function $\upsilon$ varies monotonically, though possibly not strictly monotonically. The theorem remains true when the functions $u$ and $\upsilon$ are interchanged.

*Proof.* Suppose the tangential vector $l$ makes an angle $\varphi$ with the $x$ axis at an arbitrary point of the curve $u = \text{const}$. To calculate the derivatives $u_x$ and $u_y$ as a function of the derivative $\upsilon_l$ (the derivative $u_l$ is equal to zero) by definition, we have a system consisting of Eqs (1.1) and the equations

$$\cos\varphi\, u_x + \sin\varphi\, u_y = 0, \quad \cos\varphi\, \upsilon_x + \sin\varphi\, \upsilon_y = \upsilon_l$$

Solving this system we obtain

$$u_x h(\varphi) = -\upsilon_l \sin\varphi\, |B|, \quad u_y h(\varphi) = \upsilon_l \cos\varphi\, |B|$$

Hence we have $\upsilon_l = u_n h(\varphi)/|B|$.

In the region considered the determinants $h(\varphi)$ and $|B|$ do not vanish and their signs do not change. By virtue of the choice of the curve $u = \text{const}$ the derivative $u_n$ also does not change its sign. Consequently, in the region where system (1.1) is elliptic the derivative $\upsilon_l$ does not change its sign along the chosen curve $u = \text{const}$ and, as a consequence, the function $\upsilon$ varies monotonically as one moves along the curve $u = \text{const}$.

Suppose the tangent to the curve $\upsilon = \text{const}$ makes an angle $\omega$ with the $x$ axis. Then, along the curve $\upsilon = \text{const}$, which is the boundary of the chosen region of increased or reduced values of $\upsilon$ (with respect to the curve $\upsilon = \text{const}$), we have $u_l = -\upsilon_n h(\omega)/|A|$. The monotonic variation of the function $u$ along the chosen curve $\upsilon = \text{const}$ follows from this relation. This proves the theorem.

Note also that it follows from the boundedness of the determinants $h$, $|A|$ and $|B|$ that the equalities $u_x = u_y = 0$ are only possible when $\upsilon_x = 0$ and vice versa.

The results of the theorem clearly illustrate and supplement the well-known maximum principles for the functions $u$ and $\upsilon$. In fact, if the function $u$ (or $\upsilon$) reaches an extremum at an internal point of the region in which system (1.1) is elliptic, this point will be enveloped by closed curves $u = \text{const}$ (or $\upsilon = \text{const}$), on circumventing which the function $\upsilon$ (or $u$) varies monotonically. But this is excluded for the uniquely defined functions $\upsilon$ and $u$.

A special consideration is required when the functions can be multivalued. For example, in plane subsonic gas flows, when going around closed curves of constant pressure, a change in the slope of the velocity vector by an amount which is a multiple of $2\pi$ is permissible. This is possible when there are internal stagnation points or when there are regions with closed streamlines [9].

There is a definite relationship between the monotonicity property considered above and the well-known fact that the Jacobian $J$ of the solution of elliptic system (1.1) has a constant sign. In fact, we have

$$J = u_x\upsilon_y - u_y\upsilon_x = -u_n^2 h(\varphi)/|B| = -\upsilon_n^2 h(\omega)/|A|$$

Consequently, in the region in which system (1.1) is elliptic the Jacobian $J$ does not change its sign and only vanishes when the four derivatives $u_x, \ldots, \upsilon_y$ are zero.

When analysing system (1.1) the case when $J \neq 0$ over the whole region of ellipticity is of particular interest. This enables us to invert the dependent and independent variables. The inequality $J \neq 0$ is only possible when there are no internal branching points (the absence of local-extremum points follows from the maximum principle).

As it turns out, for certain boundary-value problems, the fact that there are no branching points can be proved by analysing the curves $u$ = const and $\upsilon$ = const, emerging from the branching point considered.

We will illustrate this using the example of a problem when not more than two points correspond to each value of $\upsilon$ at the boundary of the region considered. Not less than four level curves $\upsilon$ = const emerge from the given branching point. Then, on going in a circle around the branching point between the two closest level curves, along which $\upsilon_l \geq 0$, a level curve must necessarily exist along which $\upsilon_l \leq 0$ and vice versa, respectively. These discussions are based on the fact that inside the region in which system (1.1) is elliptic, the derivatives $u_x$, $u_y$, $\upsilon_x$, $\upsilon_y$ can only be simultaneously equal to zero at isolated points but not on any section. By Theorem 1 these level curves cannot be closed on one another. Consequently, they must reach the boundary of the region, but this does not agree with the condition of the problem. Hence, the assumption that there is an internal branching point is disproved.

This situation occurs, in particular, in plane steady non-separating flow of an incompressible fluid in a closed channel formed by two non-intersecting closed convex curves. Another example, which relates to the flow around convex bodies, is given in [6].

2. Consider the inhomogeneous quasilinear elliptic system

$$a_{i1}u_x + a_{i2}u_y + b_{i1}\upsilon_x + b_{i2}\upsilon_y = c_i(x, y, u, \upsilon) \quad (i = 1,2) \tag{2.1}$$

which differs from (1.1) in having non-zero right-hand sides. For this system the conclusions drawn above regarding the monotonic change in the functions $u$ and $\upsilon$ along the curve $\upsilon$ = cost and $u$ = const, respectively, lose their meaning. In this connection those transformations of the functions $u$ and $\upsilon$ for which a system changes from being inhomogeneous to homogeneous is of some interest. Such transformations are also of interest from the point of view of converting one homogeneous system into another homogeneous system. To construct such transformations we will use the following considerations.

As we know, the homogeneous system (1.1) has a two-parametric family of solutions

$$u = a, \quad \upsilon = b \tag{2.2}$$

where $a$ and $b$ are arbitrary constants.

The converse is also true. If relations (2.2) are a solution of system (2.1) for arbitrary constants $a$ and $b$, then when $c_1 = c_2 = 0$, system (2.1) itself is homogeneous and is identical with system (1.1).

These facts suggest the following possible algorithm for converting inhomogeneous system (2.1) into homogeneous system (1.1).

Suppose system (2.1) has a two-parametric family of solutions

$$u = U(x, y, \lambda, \gamma), \quad \upsilon = V(x, y, \lambda, \gamma) \tag{2.3}$$

where $U$ and $V$ are known functions, and $\lambda$ and $\gamma$ are arbitrary constants.

For our further constructions it is more convenient to use an implicit form of writing this family of solutions, namely

$$f(x, y, u, \upsilon) = a, \quad g(x, y, u, \upsilon) = b \tag{2.4}$$

where $f$ and $g$ are known functions, and $a$ and $b$ are arbitrary constants.

The constants $a$ and $b$ need not be the same as the constants $\lambda$ and $\gamma$. They are expressed in terms of $\lambda$ and $\gamma$ using certain arbitrary functions. In other words, each family of solutions (2.3) can be written in implicit form in an unlimited number of ways.

Finally, the above discussion and a comparison of relations (2.4) and (2.2) indicate that the transformation of inhomogeneous system (2.1) into a homogeneous system can be achieved by replacing the functions $u$ and $\upsilon$ in $\alpha = f(x, y, u, \upsilon)$ and $\beta = g(x, y, u, \upsilon)$ by the functions $f$ and $g$ from (2.4).

*Theorem* 2. Suppose the inhomogeneous system (2.1) with bounded and fairly continuous functions $a_{11}, \ldots, b_{22}, c_1, c_2$ satisfies the two-parametric family of solutions (2.3), which in turn can be written in an unlimited number of ways in implicit form (2.4), and suppose also that in the region considered the Jacobian $J = f_u g_\upsilon - f_\upsilon g_u \neq 0$. Then, by replacing the functions $u$ and $\upsilon$ by the functions $\alpha = f(x, y, u, \upsilon)$ and $\beta = g(x, y, u, \upsilon)$ we can convert the inhomogeneous system (2.1) into a homogeneous system with bounded coefficients for the derivatives of $\alpha$ and $\beta$.

In fact, direct calculation of the derivatives of $u_x$, $\upsilon_x$, $u_y$, $\upsilon_y$ using the system

$$f_u u_x + f_\upsilon \upsilon_x = \alpha_x - f_x, \quad g_u u_x + g_\upsilon \upsilon_x = \beta_x - g_x$$

and a similar system with the derivatives with respect to $x$ replaced by the derivatives with respect to $y$, and substituting the values obtained into (2.1) leads to an inhomogeneous system, on the left-hand side of which the derivatives $\alpha_x$, $\alpha_y$, $\beta_x$, $\beta_y$ with bounded coefficients are collected, while on the right-hand side, in addition to the functions $c_1$ and $c_2$, there are the derivatives $f_x$, $f_y$, $g_x$, $g_y$ with coefficients. But, from the condition of the theorem, arbitrary constant values of $\alpha$ and $\beta$ are the solution of the system obtained. Consequently, the right-hand side of this system is equal to zero, while the system itself is homogeneous.

Obviously, Theorem 2 remains true if the initial system is homogeneous. In this case we are dealing with the replacement of one homogeneous system by another. In particular, since the two-parametric family of solutions (2.2) satisfies the homogeneous system (1.1), it also satisfies family (2.4) with fairly continuous arbitrary functions $f$ and $g$, which depend only on $u$ and $\upsilon$. Consequently, the non-degenerate transformation $\alpha = f(u, \upsilon)$ and $\beta = g(u, \upsilon)$, $J \neq 0$ convert system (1.1) into another homogeneous system.

We can draw the following conclusions from Theorems 1 and 2.

If the homogeneous elliptic system (1.1) (or the inhomogeneous system (2.1)) has a two-parametric family of solutions (2.4), and in the region considered the Jacobian $J = f_u g_\upsilon - f_\upsilon g_u \neq 0$, the functions $f(x, y, u, \upsilon)$ and $g(x, y, u, \upsilon)$ in the arbitrary solution of this system vary monotonically and, possibly, not strictly monotonically, along the curves $g(x, y, u, \upsilon)$ and $f(x, y, u, \upsilon) = \text{const}$. (Here, as previously, we have assumed that when passing a possible branching point of the curve $f = \text{const}$, for continuation a branch is chosen which is the limit of the chosen region of increased or decreased values of $f$.) A direct consequence of this for the function $f$ (the function $g$), which at each point of the region considered has a unique value in the $x, y$ plane, is that there are no closed curves $f = \text{const}$ ($g = \text{const}$), which lie as a whole in the region considered, and also that there are no internal extremum points of the function $f$ (the function $g$).

As it applies to homogeneous system (1.1) this indicates that the classical maximum principle, which holds for the functions $u$ and $\upsilon$, can also be extended to the function $f(x, y, u, \upsilon)$ and $g(x, y, u, \upsilon)$, which define a two-parametric family of solutions of the form (2.4) of system (1.1). We also note that for system (1.1) one can use as the functions $f$ and $g$ arbitrary fairly smooth functions which depend only on $u$ and $\upsilon$, for which the above Jacobian $J \neq 0$.

The number $N$, $0 \leqslant N \leqslant \infty$, of possible relations (2.3), and the functions $U$ and $V$ occurring in them, are determined by the form of the specific system (1.1) or (2.1) considered, and are important characteristics of the system considered. As mentioned above, each relation (2.3) can be written in implicit form (2.4) in an unlimited number of ways. Of relations (2.4) and relations (2.3) corresponding to them, those for which an analysis of the curves $f = \text{const}$ and $g = \text{const}$ enables useful conclusions to be drawn regarding the properties of the solution of the boundary-value problem for the system considered are of particular interest.

3. We will consider some applications of the results obtained above to plane and axisymmetric flows of ideal (inviscid and non-heat-conducting) incompressible fluids and gases. Vortical and potential flows are described by the following systems of equations

$$\begin{aligned} &(M^2 - 1)(p_x \cos\theta + p_y \sin\theta) + \rho q^2(-\theta_x \sin\theta + \theta_y \cos\theta) = -\mu\rho q^2 \sin\theta / y \\ &-p_x \sin\theta + p_y \cos\theta + \rho q^2(\theta_x \cos\theta + \theta_y \sin\theta) = 0 \end{aligned} \tag{3.1}$$

$$\begin{aligned} &(u^2 - c^2)u_x + u\upsilon(u_y + \upsilon_x) + (\upsilon^2 - c^2)\upsilon_y = \mu\upsilon c^2 / y \\ &u_y - \upsilon_x = 0 \end{aligned} \tag{3.2}$$

Here and henceforth $p$, $\rho$ and $s$ are the pressure, density and entropy, related by the equation of state $p = p(\rho, s)$, $q$ and $\theta$ are the modulus and slope of the velocity vector, $M$ is the Mach number, $c$ is the velocity of sound, $u$ and $\upsilon$ are the projections of the velocity vector on the $x$ and $y$ axes (in the axisymmetric case the $x$ axis coincides with the axis of symmetry), and $\mu = 0, 1$ in the plane and axisymmetric cases. In the case of an incompressible fluid $c = \infty$, $M = 0$. We will consider different types of flows in the following sequence.

*Axisymmetric vortex flows.* Two-parametric families of solutions of the form (2.3) and (2.4) are not known either for a gas or for an incompressible fluid. As a consequence, homogeneous systems of equations of the form (1.1) describing such flows are also not known.

*Plane vortex flows.* Only one two-parametric family of solutions of the form (2.3) are known both for a gas and for an incompressible fluid

$$p = \lambda = \text{const}, \quad \theta = \gamma = \text{const} \tag{3.3}$$

Consequently, the functions $p$ and $\theta$ have the property of monotonicity, according to which the function $\theta$ is monotonic along the curve $p$ = const and vice versa. For an incompressible fluid this holds for all regions of the flow, while for a gas it only holds in the subsonic region of the flow. These properties of the functions $p$ and $\theta$ were pointed out for the first time in [2] and they were actively used to analyse vortical subsonic flows in [2–7].

The family of solutions (3.3) can be written implicitly in the form (2.4) by a number of other methods, for example, in the form

$$p \cos\theta = a = \text{const}, \quad p \sin\theta = b = \text{const} \tag{3.4}$$

Consequently, the functions $f = p\cos\theta$ and $g = p\sin\theta$ also have the monotonicity property. By replacing the functions $p$ and $\theta$ by the functions $\alpha = p\cos\theta$ and $\beta = p\sin\theta$ we convert the homogeneous system (3.1) into another homogeneous system.

*Axisymmetric potential flows.* For gas flows there are two two-parametric families of solutions of the form (2.3). The first of these corresponds to the exact solution for flow from sources (sinks), uniformly distributed over the axis of symmetry with specific intensity $b$ (for $b > 0$ we have a source and for $b < 0$ we have a sink) and moving along the axis of symmetry with constant velocity $u = a$. The quantities $a$ and $b$ are also the parameters of the solution. The natural (but not unique) form of representing this flow in implicit form (2.4) is as follows:

$$u = a, \quad y\rho\upsilon = b \tag{3.5}$$

It follows from this that in arbitrary subsonic axisymmetric potential flow the functions $f = u$ and $g = y\rho\upsilon$ possess the monotonicity property, according to which $g$ is monotonic along the curve $f$ = const and vice versa. By replacing the functions $u$ and $\upsilon$ by the functions $\alpha = u$ and $\beta = y\rho\upsilon$ we convert the inhomogeneous system (3.2) into the following homogeneous system

$$(1-M^2)\alpha_x - \frac{u\upsilon}{y\rho c^2}\beta_x + \frac{c^2-\upsilon^2}{y\rho c^2}\beta_y = 0$$

$$\frac{u\upsilon}{c^2}\alpha_x - \frac{c^2-\upsilon^2}{c^2}\alpha_y + \frac{\beta_x}{y\rho} = 0$$

where the Jacobian of the transition for $M < 1$ does not vanish. The monotonicity property of the functions $f = u$ and $g = y\upsilon$ for asymptotic transonic axisymmetric equations was pointed out earlier in [8]. The second two-parametric family of solutions corresponds to flow from a source (sink) of intensity $b$, placed on the axis of symmetry at the point $x = x_0$; the parameters of the flow are $x_0$ and $b$. One of the possible forms of writing it in the implicit form (2.4) is

$$f = x - \frac{yu}{\upsilon} = x_0, \quad g = \frac{\rho y^2(u^2+\upsilon^2)^{3/2}}{\upsilon^2} = b \tag{3.6}$$

Consequently, the functions $f$ and $g$ possess the monotonicity property in the subsonic region; by replacing the functions $u$ and $\upsilon$ by the functions $\alpha = f$ and $\beta = g$ we can also convert the inhomogeneous system (3.2) into a homogeneous system, which we will not give here because of its length.

In the case of an incompressible fluid we can add to the above results an unlimited set of two-parametric families of solutions of the form (3.2), corresponding to the interaction between a uniform oncoming flow and an arbitrary number of sources, sinks and doublets, situated on the axis of symmetry. We will confine ourselves to considering the interaction between a uniform flow $u = a$ and a source (sink) of intensity $b$ situated at the origin of coordinates. The potential of this flow has the form

$$\varphi = ax - b(x^2 + y^2)^{-1/2}$$

Taking into account the fact that $\varphi_x = u$, $\varphi_y = \upsilon$, the two-parametric family of solutions in the form (2.4) can be written as follows:

$$f = u - \frac{\upsilon x}{y} = a, \quad g = \frac{\upsilon (x^2 + y^2)^{3/2}}{y} \tag{3.7}$$

Consequently, the functions $f$ and $g$ have the monotonicity property, and by replacing the functions $u$ and $\upsilon$ by the functions $\alpha = f$ and $\beta = g$ we can convert the inhomogeneous system (3.2) into the following homogeneous system

$$(x^2 + y^2)^{3/2}\alpha_x + x\beta_x + y\beta_y = 0, \ (x^2 + y^2)^{3/2}\alpha_y - y\beta_x + x\beta_y = 0$$

To conclude this section on axisymmetric potential flows, we note that the homogeneous systems derived above for these flows, like other homogeneous systems which can be constructed using this algorithm, are also of some independent interest, unrelated to the monotonicity properties.

*Plane potential flows.* For gas flows we can propose a four-parametric family of solutions corresponding to the superposition of a source (sink) of intensity $b$ and a potential vortex of intensity $a$ with a common centre situated at the point $x = x_0$, $y = y_0$. The parameters are the quantities $a$, $b$, $x_0$ and $y_0$. By fixing any two parameters or two combinations of these parameters we can construct a fairly large family of two-parametric solutions of the form (2.3) and (2.4).

We will confine ourselves to two cases.

Suppose the parameters $x_0$ and $y_0$ are fixed. The solution in the form (2.4) can then be written as follows:

$$f = \rho(u(x - x_0) + \upsilon(y - y_0)) = a, \quad g = \upsilon(x - x_0) - u(y - y_0) = b \tag{3.8}$$

Consequently, for subsonic plane potential flows the functions $f$ and $g$ possess the monotonicity property, and by making the replacement $\alpha = f$, $\beta = g$ we can convert the homogeneous system (3.2) into another homogeneous system

$$((x - x_0)(M^2 - 1) - \upsilon\beta / c^2)\alpha_x + ((y - y_0)(M^2 - 1) + u\beta / c^2)\alpha_y -$$

$$-\rho(M^2 - 1)((y - y_0)\beta_x - (x - x_0)\beta_y) = 0$$

$$(y - y_0)\alpha_x - (x - x_0)\alpha_y - \rho((x - x_0)(M^2 - 1) - \upsilon\beta / c^2)\beta_x - \rho((y - y_0)(M^2 - 1) + u\beta / c^2)\times \beta_y = 0$$

For an incompressible fluid we obtain, after some reduction

$$\alpha_x + \beta_y = 0, \quad \alpha_y - \beta_x = 0$$

In the next example we fix the parameters $a$ and $b$, and the family of solutions in the form (2.4) can be written as follows:

$$f = x - \frac{a\cos\theta}{\rho q} - \frac{b\sin\theta}{q} = x_0, \quad g = y - \frac{a\sin\theta}{\rho q} + \frac{b\cos\theta}{q} = y_0 \tag{3.9}$$

Consequently, for the flows considered the functions $f$ and $g$ possess the monotonicity property, and by making the replacement $\alpha = f$, $\beta = g$ we can convert the homogeneous system (3.2) into another homogeneous system. As might have been expected, each of the equations (3.2) converts into a very complex equation. But, after some reduction, the system can be reduced to the following fairly compact system

$$(2 - M^2 \sin^2 2\theta)(\alpha_x - \beta_y) - 2M^2 \sin 2\theta(\alpha_y \sin^2\theta - \beta_x \cos^2\theta) = 0$$
$$(1 - M^2 \sin^2\theta)\alpha_y + (1 - M^2\cos^2\theta)\beta_x = 0$$

where, by (3.9), the functions $M$ and $\theta$ depend on $x$, $y$, $\alpha$, $\beta$, $a$ and $b$.

For an incompressible fluid, the system has the following form

$$\alpha_x - \beta_y = 0, \quad \alpha_y + \beta_x = 0$$

It is characteristic that in both of the previous examples it is not obvious if, dispensing with the properties of the two-parametric families of solutions of the form (2.4) and replacing the functions $u$ and $\upsilon$ by the functions $\alpha$ and $\beta$, which differ considerably from one another, in the case of an incompressible fluid, convert the Cauchy–Riemann system (3.2) into other Cauchy–Riemann systems.

In addition to the situations considered, in the case of an incompressible fluid we can construct an unlimited set of two-parametric families of solutions of the form (2.3), corresponding to the interaction between a uniform free stream and an arbitrary number of sources, sinks, doublets and potential vortices, the centres of which are situated at arbitrary points of the $x$, $y$ plane.

We will confine ourselves to considering the interaction between a uniform horizontal free stream with velocity $u = a$ and the flow from a source (sink) of intensity $b$, situated at the origin of coordinates. The potential of the resultant flow has the form

$$\varphi = ax + b\ln(x^2 + y^2)/2$$

A two-parametric family of solutions can be written in implicit form (2.4) as follows:

$$f = u - \upsilon x/y = a, \quad g = \upsilon(x^2 + y^2)/y = b \tag{3.10}$$

Consequently, the functions $f$ and $g$ possess the monotonicity property, and by making the replacement $\alpha = f$, $\beta = g$ we can convert the homogeneous system (3.2) into the following homogeneous system

$$(x^2 + y^2)\alpha_x + x\beta_x + y\beta_y = 0$$
$$(x^2 + y^2)\alpha_y - y\beta_x + x\beta_y = 0$$

The functions $f$ and $g$ constructed above in the ellipticity regions possess the monotonicity property, according to which the function $g$ varies monotonically along the $f$ = const curve and vice versa. It follows from this that at an arbitrary internal point of this region, in which $f$ and $g$ are uniquely defined as functions of the coordinates of this point, the functions may not reach its local extremum.

Otherwise, in a fairly small neighbourhood of this point closed level curves of the function considered should in fact exist, which contradicts the monotonicity property.

Isolated points at which one or both of the functions considered is multi-valued are an exception. In this case closed level curves are possible which encompass this point or which pass through it and, consequently, a local extremum of one or both of the functions considered is possible at this point.

As it applies to the functions constructed above, we can distinguish three types of such points: internal stagnation points, which are possible, for example, when jets collide (at these points the function $\theta$ is multivalued, which is important for $f$ and $g$ in relations (3.3), (3.4), (3.6) and (3.9)); a vortex centre, at which $p = 0$ (this situation is possible in an incompressible fluid; at this point the function $\theta$ is also multivalued; this situation is important for $f$ and $g$ in relations (3.3), (3.5)–(3.8) and (3.10)); a point with coordinates $x = y = 0$, at which the function $x/y$ is multivalued (this is only important for the functions $f$ and $g$ from (3.7) and (3.10)).

In conclusion we note that the extremal properties of the functions $p$ and $\theta$ for plane vortex flows were established in [2]. Moreover, the extremal properties of the functions $u$, $\upsilon$, $q$ and $\theta$ for plane potential flows are well known from classical courses in mathematical physics.

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